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Recurrence Time Statistics in Chaotic Dynamics. I. Discrete Time Maps

V. Balakrishnan,[1,2] G. Nicolis,[2] and C. Nicolis[3]

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The dynamics of transitions between the cells of a finite-phase-space partition in a variety of systems giving rise to chaotic behavior is analyzed, with special emphasis on the statistics of recurrence times. In the case of one-dimensional piecewise Markov maps the recurrence problem is cast into a renewal process. In the presence of intermittency, transitions between cells define a non-Markovian, non-renewal process reflected in the presence of power-law probability distributions and of divergent variances and mean values.

KEY WORDS: Recurrence time; escape time; Markov partition; fully developed chaos; intermittent chaos.

1. INTRODUCTION

Ever since the time of Boltzmann and Poincaré, the recurrence of states of a dynamical system has been recognized both as one of the principal signatures of its deterministic origin and as a challenge to be met in building a consistent statistical theory of the underlying system compatible with the fundamental laws of physics.[1]

In its classical version, Poincaré's recurrence theorem refers to a one-parameter family F^t of one-to-one measure-preserving transformations. It states that if C is a subset of the phase space Γ such that $\mu(C) > 0$ [μ is a completely additive measure with $\mu(\Gamma) = 1$], then for almost every point $P \in C$ there exists a sufficiently large t such that $F^t P \in C$. By discretizing

[1] Department of Physics, Indian Institute of Technology, Madras 600 036, India.
[2] Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, C.P. 231, 1050 Brussels, Belgium.
[3] Institut Royal Météorologique de Belgique, 1180 Brussels, Belgium.

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time in slices of duration τ and further assuming that the transformation F^{τ} is metrically transitive (ergodic), one can then derive the following expression for the mean recurrence time,

$$\langle \theta_\tau \rangle = \tau/\mu(C) \tag{1.1}$$

Unfortunately, for realistic many-body systems it is extremely hard to decide whether a given Hamiltonian will give rise to metrically transitive transformations. Moreover, Eq. (1.1) suffers from the disadvantage that in the limit of continuous time, $\tau \to 0$, it gives the trivial (and wrong) result 0.

With the advent of modern chaos theory it has become possible to set up simple-looking, low-dimensional models giving rise to complex behavior that emulates, to a large extent, the behavior of real world, multivariate systems.[2] Such models constitute ideal case studies for revisiting the formalism of statistical mechanics, for verifying otherwise elusive conjectures, and for understanding detailed mechanisms to a degree that cannot be attained in the many-particle systems usually considered in statistical mechanics.[3-5] Furthermore, dissipative chaos modeling complex behavior at the macroscopic level, from aperiodic oscillations in chemistry and electronics to turbulence and some aspects of atmospheric and climatic variability, can also be approached in probabilistic terms. This allows one to bypass the fundamental limitations imposed by sensitivity to initial conditions via probabilistic rather than deterministic predictions, and to evaluate various quantities of practical interest as statistical averages. The investigation of recurrence time statistics and related properties of a class of chaotic, dissipative, noninvertible maps is the principal objective of this work. The case of continuous-time systems undergoing invertible dynamics will be considered in a separate publication.

The general formulation is laid down in Section 2. The starting point is the Frobenius–Perron (FP) equation for the probability density.[3,4] Since recurrence can only be formulated (except for a trivial case) in terms of finite volumes in phase space, the question immediately arises whether the FP equation can generate a well-defined random process once projected onto a phase space partition. This question admits a simple answer for fully developed chaotic systems of the tent map type, for which the projected dynamics reduces to a Markov chain.[6] In Section 3 the explicit form of the transition probability matrix is obtained for this system, and a full theory of recurrence time statistics is worked out and compared successfully with the results of numerical simulation.

Section 4 is devoted to two prototype models of intermittent chaos: the continuous, symmetric cusp map and its discontinuous, antisymmetric counterpart. The process generated by the deterministic dynamics on the phase space partition is now shown to be highly non-Markovian. Nevertheless, a detailed theory of recurrence statistics can again be worked out and compared with the results of numerical simulations. In Section 5 a number of related statistical properties such as sojourn, escape, and first passage times are studied. The intermittency arising from the marginally stable fixed points of the maps is shown to have interesting and subtle effects on the distributions concerned. Our main conclusions are summarized in Section 6.

2. GENERAL FORMULATION

Consider the discrete-time dynamical system

$$\mathbf{X}_{n+1} = \mathbf{f}(\mathbf{X}_n, \mu), \qquad n = 0, 1,... \tag{2.1}$$

where $\mathbf{f}$ is the evolution law and μ the control parameter. It will be assumed that under $\mathbf{f}$ the state variables $\mathbf{X}$ remain confined to a finite, invariant part Γ of phase space. In what follows we shall be interested in evolution laws and parameter ranges for which the dynamics is chaotic. As is well known, Eq. (2.1) generates an evolution equation for the probability density $\rho_n(\mathbf{X})$, $\mathbf{X} \in \Gamma$, namely, the FP equation

$$\rho_{n+1}(\mathbf{X}) = \int_\Gamma d\mathbf{Y}\, \delta(\mathbf{X} - \mathbf{f}(\mathbf{Y}, \mu))\, \rho_n(\mathbf{Y}) \tag{2.2}$$

To formulate the problem of recurrence, we consider a finite cell $C \in \Gamma$ and assume that the evolution is started at a point $\mathbf{X}_0 \in \Gamma$. As the evolution proceeds, the representative point will in general escape from C, but, unless it is part of an exceptional (e.g., periodic) orbit or the system has poor ergodic properties, it will be reinjected repeatedly into C. Let

$$F(C, 0; C, n) = \text{Prob}(\mathbf{X}_0 \in C, \mathbf{X}_1 \notin C,..., \mathbf{X}_{n-1} \notin C, \mathbf{X}_n \in C) \tag{2.3}$$

be the normalized probability of the first return of the representative point to the cell C at time n. The mean recurrence time is then

$$\langle n_{cc} \rangle = \sum_{n=1}^{\infty} nF(C, 0; C, n) \tag{2.4}$$

Higher order moments of the recurrence time distribution can be defined similarly.

Since the deterministic evolution law implies that the members of an initial Gibbs ensemble are propagated in time by a δ-function type transition probability, $F(C, 0; C, n)$ can also be expressed as

$$F(C, 0; C, n) = \int_C d\mathbf{X}_0\, \rho(\mathbf{X}_0) \int_{\bar{C}} d\mathbf{X}_1 \cdots \int_{\bar{C}} d\mathbf{X}_{n-1} \int_C d\mathbf{X}_n \times \prod_{r=1}^{n} \delta(\mathbf{X}_r - \mathbf{f}^{(r)}(\mathbf{X}_0, \mu)) \Big/ \int_C d\mathbf{X}_0\, \rho(\mathbf{X}_0) \tag{2.5}$$

where ρ is the invariant density and $\bar{C}$ is the complement of C in Γ. In practice, the explicit evaluation of (2.5) may not be feasible. For this reason we resort to the statistical description afforded by the FP equation (2.2). As mentioned in the Introduction, the chief difficulty to be met here is the characterization of the process of transition across the boundaries of the cell. As a preliminary toward this goal, we introduce a partition of the phase space into K nonoverlapping cells $\{C_j\}$ (of which the cell C referred to above is just one member), and define a projection operator $\mathbf{E}$ onto this partition[6-8] according to

$$\mathbf{P}_n = \mathbf{E}\rho_n(\mathbf{X}) = \sum_{j=1}^{K} \mu^{-1}(C_j) \int_{C_j} d\mathbf{Y}\, \rho_n(\mathbf{Y})\, \chi(\mathbf{X}) \tag{2.6}$$

where $\mathbf{P}_n = (P_n(1),..., P_n(K))$ is the probability vector induced by $\rho_n(\mathbf{X})$ in the partition and $\chi = (\chi_1,..., \chi_K)$ denotes the characteristic function of the partition, i.e., $\chi_j(\mathbf{X}) = 1$ if $\mathbf{X} \in C_j$, 0, if $\mathbf{X} \notin C_j$. Applying the operator $\mathbf{E}$ on both sides of Eq. (2.2) we obtain

$$\mathbf{P}_{n+1} = \mathbf{E} \int d\mathbf{Y}\, \delta(\mathbf{X} - \mathbf{f}(\mathbf{Y}, \mu))\, \rho_n(\mathbf{Y}) \tag{2.7}$$

The right-hand side cannot in general be expected to reduce to the form of a time-independent transition matrix acting on the probability vector $\mathbf{P}_n$. In what follows we consider two representative cases in which the foregoing partition can nevertheless be explicitly implemented and a recurrence time statistics generated, referring, respectively, to fully developed chaos and intermittent chaos.

3. FULLY DEVELOPED CHAOS: A MARKOVIAN CASE

The representative illustration we consider is the one-dimensional endomorphism (the tent map at fully developed chaos)

$$X_{n+1} = f(X_n) = 1 - |2X_n - 1|, \qquad X_n \in [0, 1] \tag{3.1}$$

whose invariant density is uniform, $\rho(X) = 1$. Markov partitions of this map constitute special cases of a general theory of Markov coarse graining of piecewise linear maps.[6] In the present instance, we are interested in a partitioning of the unit interval (the phase space Γ) into K *equal* segments (cells) such that $C_j = [(j-1)/K, j/K)$, $j = 1,..., K$. Then, denoting the n-step probability of a transition from cell C_j to cell C_k by $P(j, 0; k, n)$, we have

$$P(j, 0; k, n) = \int_{C_j} dY \int_{C_k} dX\, \delta(X - f^{(n)}(Y))\, \rho(Y) \bigg/ \int_{C_j} dY\, \rho(Y) \tag{3.2}$$

Since the invariant density is a constant in the present case, the measure $\mu(C_j) = \int_{C_j} dY\, \rho(Y) = 1/K$. On the other hand, performing the delta function by integrating over the variable X reduces the domain of integration over Y to the set $C_j \cap C_k^{-n}$, C_k^{-n} being the nth pre-image of C_k. The value of the resulting integral in Eq. (3.2) can be deduced in the following simple geometric way: Partitioning the unit square into K^2 square cells of side $1/K$, we plot $f^{(n)}(Y)$ versus Y. Then $P(j, 0; k, n)$ is simply K times the magnitude of the support in Y of the function $f^{(n)}(Y)$ in the cell labeled by the coordinates (j, k).

The one-step transition probability $P(j, 0; k, 1)$, which we may write in suggestive notation as the jk-element of a time-independent, $(K \cdot K)$ transition matrix $\mathbf{W}$, is found from Eq. (3.2) to be given by

$$W_{jk} = K \int_{(j-1)/K}^{j/K} dY \int_{(k-1)/K}^{k/K} dX\, \delta(X - 1 + |2Y - 1|) \tag{3.3}$$

This object can be computed explicitly (the geometrical interpretation mentioned earlier is an easy way of doing so!) We find

$$\begin{aligned} &W_{jk} = \tfrac{1}{2}(\delta_{2j-1,k} + \delta_{2j,k}), \qquad 1 \leqslant j \leqslant [K/2] \\ &W_{jk} = W_{K-j+1,k}, \qquad W_{(K+1)/2,k} = \delta_{K,k} \qquad (K \text{ odd}) \end{aligned} \tag{3.4}$$

where $[K/2] = K/2$ for even K, and $(K-1)/2$ for odd K. We note that $\mathbf{W}$ is a doubly stochastic matrix.

The foregoing partitioning of the unit interval into K equal parts turns out to be a Markov partitioning for every K. This follows from the general theory[6] once we observe that all the boundary points j/K $(j = 1, 2,..., K-1)$ of the partition either belong to (unstable) periodic orbits of the map or are preimages of periodic points (but are not, themselves on periodic orbits). This has the immediate consequence that $\mathbf{W}$ is the transition matrix of a stationary, K-state Markov chain. In other words,

$$P(j, 0; k, n) = (\mathbf{W}^n)_{jk} \tag{3.5}$$

where $n = 1, 2,...;\ j, k = 1,..., K$. It is straightforward to verify that the Markov chain is irreducible and ergodic.[9] As **W** is doubly stochastic, the limiting distribution $\lim_{n \to \infty} P(j, 0; k, n) = 1/K$ is uniform, as it ought to be for consistency [we used $\rho(X) = 1$ as an input].

The eigenvalues of **W** govern the coarse-grained dynamics of the system. The largest eigenvalue, 1, is nondegenerate and corresponds, of course, to the invariant distribution $\mu(C_j) = 1/K$. Of the remaining $K-1$ eigenvalues all of which have moduli less than unity, as many have modulus 1/2 as there are points j/K ($j = 1,..., K-1$) that belong to periodic orbits of the map. The rest of the eigenvalues (as many of the set $\{j/K\}$ as are preimages of periodic points) are equal to zero. For example, when $K = 10$ we have $\lambda_1 = 1$, $\lambda_{2,3} = \pm i/2$ (corresponding to the period-2 cycle formed by the boundary points 4/10 and 8/10), and $\lambda_4 = \cdots = \lambda_{10} = 0$. The relaxation of an initial distribution to the invariant distribution therefore occurs with a characteristic time $1/\ln 2$, except for the case $K = 2^N$. We have then a Bernoulli partition: all the boundary points are preimages of the (unstable) fixed point 0, and the (coarse-grained) invariant distribution is attained in a single time step.

Our interest here is in the recurrence time statistics of the coarse-grained dynamics. This is specified by the set $\{F(j, 0; j, n) \mid 1 \leqslant j \leqslant K\}$, where $F(j, 0; j, n)$ is the probability of the first return to cell C_j at time n as defined in Eq. (2.3). As the dynamics is that of a Markov chain, the successive waiting times are mutually independent random variables and we have the renewal equation[9]

$$P(j, 0; j, n) = \sum_{n'=1}^{n} F(j, 0; j, n')\, P(j, 0; j, n-n') \tag{3.6}$$

The convolution sum appearing in the right hand side can be handled conveniently in terms of the generating functions

$$\tilde{P}_{jk}(s) = \sum_{n=1}^{\infty} P(j, 0; k, n)\, s^n, \qquad \tilde{F}_{jj}(s) = \sum_{n=1}^{\infty} F(j, 0; j, n)\, s^n \tag{3.7}$$

We obtain

$$\tilde{F}_{jj}(s) = \tilde{P}_{jj}(s)/[1 + \tilde{P}_{jj}(s)] \tag{3.8}$$

As a return to any cell j is a sure event, $\tilde{F}_{jj}(1) = 1$. Using the Markov property (3.5), we have

$$\tilde{P}_{jj}(s) = \sum_{n=1}^{\infty} (\mathbf{W}^n)_{jj}\, s^n = [s\mathbf{W}(1 - s\mathbf{W})^{-1}]_{jj} \tag{3.9}$$

in obvious notation.

The mean time of recurrence in cell C_j is given by $\langle n_{jj} \rangle = \tilde{F}'_{jj}(1)$. As **W** is doubly stochastic with a nondegenerate eigenvalue 1, we can show by introducing its spectral decomposition and by isolating the part pertaining to the leading eigenvalue that

$$\tilde{P}_{jj}(s) = \frac{s}{K(1-s)} + \tilde{R}_{jj}(s) \tag{3.10}$$

in the neighborhood of $s=1$, where $\tilde{R}_{jj}(s)$ is regular at $s=1$. When substituted in Eq. (3.8), this suffices to establish that

$$\langle n_{jj} \rangle = K = \mu^{-1}(C_j), \qquad \forall j \tag{3.11}$$

in accord with the ergodic theorem[9] and Eq. (1.1). The variance of the recurrence time, however, depends on the cell index j, being given by

$$\mathrm{Var}\, n_{jj} = F''_{jj}(1) + F'_{jj}(1) - F'^2_{jj}(1) = K^2[1 + 2\tilde{R}_{jj}(1)] - K \tag{3.12}$$

As a check on these predictions, we compare the results in the simplest nontrivial case, $K=3$, with those of numerical experiments. As the point 1/3 is a preeimage of the point 2/3 (which is a fixed point of the map), the eigenvalues of **W** are 1, $-1/2$, and 0 in this instance. Using Eqs. (3.9) and (3.8) to compute $\tilde{P}_{jj}(s)$ and $\tilde{F}_{jj}(s)$ and inverting the latter transform, we find straightforwardly the following recurrence time distributions. For cell $C_1 = [0, 1/3)$,

$$F(1, 0; 1, n) = [1 - (-1)^n]\, 2^{-(n+3)/2} \tag{3.13}$$

For cells $C_2 = [1/3, 2/3)$ and $C_3 = [2/3, 1]$,

$$F(2, 0; 2, n) = F(3, 0; 3, n) = (1 - \delta_{n,1})\, 2^{1-n} \tag{3.14}$$

The gaps in the distribution of Eq. (3.13) for even values of n should be noted. The mean values are $\langle n_{11} \rangle = \langle n_{22} \rangle = \langle n_{33} \rangle = 3$, in agreement with the general result of Eq. (3.11). The variances, however, differ considerably: we find $\mathrm{Var}\, n_{11} = 8$, while $\mathrm{Var}\, n_{22} = \mathrm{Var}\, n_{33} = 2$, the large scatter in n_{11} being caused by the slower decay of the distribution $F(1, 0; 1, n)$ with increasing n. Furthermore, their magnitude is seen to be comparable to that of the mean values. Figure 1 depicts the numerically evaluated recurrence time distributions in full agreement with the foregoing theoretical values. The agreement extends to other values of K as well.

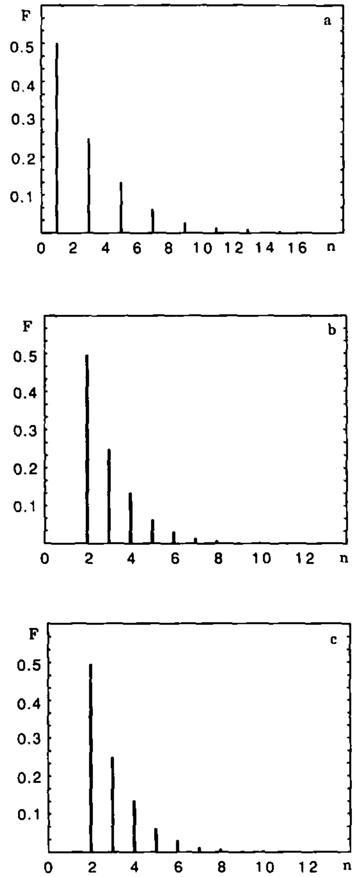


Fig. 1. Recurrence time distributions for the tent map, Eq. (3.1), in (a) cell $C_1 = [0, 1/3)$, (b) $C_2 = [1/3, 2/3)$, and (c) $C_3 =]2/3, 1]$, as obtained from 9000 recurrences. The mean value $\langle n \rangle$ in each cell is found to be equal to 3, whereas the variances are 7.65, 2.0, and 2.0 respectively.

4. INTERMITTENT CHAOS: NON-MARKOVIAN CASES

In the example studied in Section 3, we were aided substantially by the fact that the coarse-grained dynamics reduced to that of a finite Markov chain for every K. We turn now to two prototypical models that display intermittent chaos, for which even the simplest partitioning of the phase space leads to intricate non-Markovian properties. In order to probe the effects of the curvature of the map function, of a nonconstant invariant density, of the intermittency occasioned by the tangency of the map to the bissectrix, and of two competing marginally stable fixed points, we consider in this Section the following two maps of the interval $[-1, 1]$ onto itself:

(i) The symmetric, continuous, square-root cusp map

$$S: \quad f_S(X) = 1 - 2\,|X|^{1/2} \tag{4.1}$$

The map is tangent to the bissectrix at the marginally stable fixed point $X = -1$, and there is an unstable fixed point at $3 - 2\sqrt{2}$.

(ii) The antisymmetric, discontinuous counterpart of the foregoing,

$$A: \quad f_A(X) = (2\,|X|^{1/2} - 1)\,\mathrm{sgn}\,X \tag{4.2}$$

which has marginally stable fixed points at both ends, $X = -1$ and $X = +1$. Both systems are ergodic, with unique, smooth invariant densities in $[-1, 1]$ given respectively by

$$\begin{aligned} \rho_S(X) &= (1 - X)/2 \\ \rho_A(X) &= \text{const} = 1/2 \end{aligned} \tag{4.3}$$

Let us partition the phase space into two cells of equal size, $C_1 \equiv L = [-1, 0)$ and $C_2 \equiv R = [0, 1]$. This partition no longer generates a metric Markov chain, although it remains Markov in a topological sense. It is easy to compute the one-step transition probabilities $W_{jk} \equiv P(j, 0; k, 1)$ $(j, k = 1, 2$ or $L, R)$, defined as in Eq. (3.2) with $n = 1$. We find

$$W_S = \tfrac{1}{16}\begin{pmatrix} 13 & 3 \\ 9 & 7 \end{pmatrix}, \qquad W_A = \tfrac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{4.4}$$

for the symmetric and antisymmetric maps, respectively. W_S is not doubly stochastic, since $\rho_S \neq \text{const}$. On the other hand, W_A reflects the $L \leftrightarrow R$ symmetry of the discontinuous map. The crucial point is that the cell dynamics is no longer Markovian. This may be verified, for instance, by comparing

$$W_S^2 = \tfrac{1}{128}\begin{pmatrix} 98 & 30 \\ 90 & 38 \end{pmatrix}, \qquad W_A^2 = \tfrac{1}{32}\begin{pmatrix} 20 & 12 \\ 12 & 20 \end{pmatrix} \tag{4.5}$$

with the actual two-step probabilities

$$P_S(n=2) = \tfrac{1}{128}\begin{pmatrix} 101 & 27 \\ 81 & 47 \end{pmatrix}, \qquad P_A(n=2) = \tfrac{1}{32}\begin{pmatrix} 32 & 9 \\ 9 & 23 \end{pmatrix} \tag{4.6}$$

respectively. Although the boundary point 0 is a preimage of a fixed point for the maps S and A, the remarks made in Section 3 regarding Markov partitioning do not apply here, as the maps are not piecewise linear. Thus it is the curvature of the maps that is responsible for the non-Markov nature of the dynamics in terms of the states L and R.

The explicit evaluation of the n-step probabilities $P(j, 0; k, n)$ for arbitrary n is quite complicated for the maps under study. Moreover, the renewal equation (3.6) does *not* hold good here: not only is the coarse-grained dynamics non-Markovian, *it is not even a renewal process* in the sense that the recurrence times are no longer mutually independent random variables, owing to the memory implied by the deterministic dynamics. Nevertheless, it turns out to be possible to compute the recurrence time distributions $F(j, 0; j, n)$ directly from the definition (2.5). The details are given in the Appendix, and the results are as follows.

Consider the monotonically decreasing sequence $\{u_r\}$ defined by the recursion relation

$$u_r = u_{r-1}(1 - u_{r-1}), \qquad u_0 = 1/4 \tag{4.7}$$

This is the logistic map at parameter value 1, at which the fixed point $u = 0$ becomes marginally stable. Then, for the discontinuous map A, we find [see Eq. (A.8)].

$$\left.\begin{aligned} F_A(L, 0; L, n) &= F_A(R, 0; R, n) \\ &= 4(u_{n-2}^2 - u_{n-1}^2) \\ &= 4(u_{n-2} - 2u_{n-1} + u_n) \end{aligned}\right\} \tag{4.8}$$

With $u_{-1} = 1/2$, Eq. (4.8) remains valid for $n = 1$ as well. For the continuous cusp map S, there is no $L \leftrightarrow R$ symmetry in the coarsegrained dynamics. We get

$$F_S(R, 0; R, n) = 16(u_{n-1} - 2u_n + u_{n+1}), \qquad n \geqslant 1 \tag{4.9}$$

On the other hand, for recurrence in cell L we get [see Eqs. (A.16), (A.17)]

$$F_S(L, 0; L, n) = \frac{16\bar{\lambda}}{3}(-1)^n (v_{n+2} + v_{n+1} - v_n - v_{n-1}), \qquad n \geqslant 1 \tag{4.10}$$

where $\bar{\lambda} = \sqrt{2} - 1$, and $\{v_r\}$ is given by

$$v_r = -\bar{\lambda} v_{r-1}(1 - v_{r-1}), \qquad v_0 = 1/2 \tag{4.11}$$

The logistic map thus appears once again, but at a parameter value that is less than unity in magnitude.

Using the simple closed expressions in Eqs. (4.8)–(4.10), we may verify easily that all the distributions obtained above are normalized to unity, i.e., the recurrences concerned are sure events. The mean recurrence times are also obtained readily. For the discontinuous map A, we find

$$\langle n_{LL} \rangle_A = \langle n_{RR} \rangle_A = 2 \tag{4.12}$$

For the continuous map S, we get

$$\langle n_{RR} \rangle_S = 4, \qquad \langle n_{LL} \rangle_S = 4/3 \tag{4.13}$$

These mean values, fully corroborated by numerical experiments, are precisely the reciprocals of the respective measures $\mu_A(R) = \mu_A(L)$, $\mu_S(R)$, and $\mu_S(L)$. This is a consequence of the ergodicity of the dynamics, notwithstanding its highly correlated, non-Markovian character. The effects of intermittency are revealed in the higher moments of the recurrence times. To understand this we require the asymptotic $(n \to \infty)$ behavior of the corresponding distributions. It is straightforward to establish[10] that, for very large values of n,

$$u_n = n^{-1} - n^{-2} \ln n + O(n^{-3}) \tag{4.14}$$

while

$$|v_n| = \text{const} \cdot e^{-\mu n} + O(e^{-2\mu n}) \tag{4.15}$$

where $\mu = \ln(\sqrt{2} + 1)$. Therefore we have the leading asymptotic behaviors

$$F_A(L, 0; L, n) = F_A(R, 0; R, n) \sim n^{-3} \tag{4.16}$$

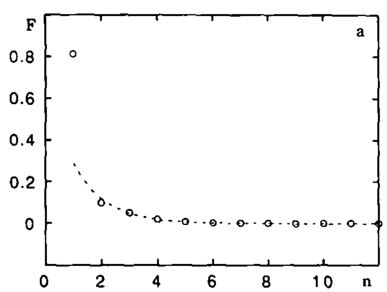
$$F_S(R, 0; R, n) \sim n^{-3}, \qquad F_S(L, 0; L, n) \sim e^{-\mu n} \tag{4.17}$$

The second moments $\langle n_{LL}^2 \rangle_A = \langle n_{RR}^2 \rangle_A$ and $\langle n_{RR}^2 \rangle_S$ therefore diverge (logarithmically, like $\sum_1^\infty u_n$), while all the moments $\langle n_{LL}^q \rangle_S$ $(q = 2, 3, \ldots)$ are finite. In particular, we find

$$\begin{aligned} (\operatorname{Var} n_{LL})_S &= \tfrac{64}{3} \bar{\lambda} \sum_{r=0}^{\infty} |v_r| + \tfrac{5}{12} - \sqrt{2} \\ &\approx 0.715 \end{aligned} \tag{4.18}$$

In general, numerical simulation involving the maps S and A is a difficult task: The presence of marginally stable fixed points makes the attainment of the invariant distribution an extremely slow process, and the results generally turn out to be sensitively dependent on the numerical accuracy adopted. The extraction of reliable results thus requires considerable care in the choice of initial values, accuracy, and duration of runs. The logarithmic divergences of $\langle n_{LL}^2 \rangle_A$ and $\langle n_{RR}^2 \rangle_S$ found above show up in numerical experiments as slow but systematic increases (rather than convergence to definite limiting values) as the number of time steps is increased. Moreover, the values obtained are larger, the higher the numerical resolution. The marginal stability at $X = -1$ also affects $(\mathrm{Var}\, n_{LL})_S$ (which is finite), as does the oscillatory convergence of the sequence $\{z_r\}$ that determines $F_S(L, 0; L, n)$ [cf. Eq. (A.15)]: we find $(\mathrm{Var}\, n_{LL})_S \approx 0.734$ for 20,000 passages, which is not too different from the exact value 0.715 obtained in Eq. (4.18), given the mitigating factors just cited. For the same number of passages one finds $(\mathrm{Var}\, n_{RR})_S \approx 53$, hinting that the divergence stipulated by the theory may indeed be present.

Figure 2 depicts the numerically computed distributions $F_S(R, 0; R, n)$ and $F_S(L, 0; L, n)$ and their fitting by, respectively, a power law and an exponential for $n \geqslant 5$. Again, the agreement with the theoretical results is rather satisfactory.



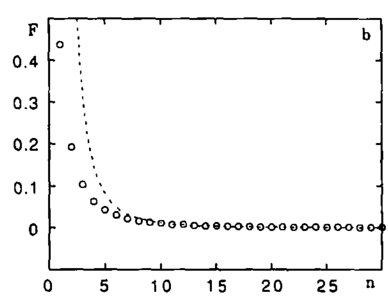


Fig. 2. Recurrence time distributions for the symmetric cusp map, Eq. (4.1), in (a) cell $C_1 = [-1, 0]$, and (b) $C_2 = (0, 1]$, as obtained from 100,000 recurrences (open dots). Dashed line represents a best fit with (a) $K_1 \exp[-\ln(\sqrt{2} + 1)\, n]$ and (b) $K_2 n^{-3}$.

5. OTHER STATISTICS: SOJOURN, ESCAPE AND FIRST PASSAGE TIMES

Having seen how recurrence times are distributed, we turn now to the behavior of related quantities such as sojourn or halting times, escape times, and first passage times in the coarse-grained dynamics of the prototypical maps considered in Sections 3 and 4.

5.1. Sojourn Times

The probability that the sojourn or halt in cell C_j persists till time n is given by

$$H(j, 0; j, n) = \mu^{-1}(C_j) \int_{C_j} dX_0 \cdots \int_{C_j} dX_n \, \rho(X_0) \times \prod_{r=1}^{n} \delta(X_r - f(X_{r-1})) \tag{5.1}$$

For a Markov partitioning this simplifies considerably, as one might expect: if W is the corresponding transition matrix, we have

$$H(j, 0; j, n) = (W_{jj})^n \tag{5.2}$$

In the case of the tent map with a K-cell partitioning of the unit interval, $\mathbf{W}$ is given by Eq. (3.4). When K is a multiple of 3, the fixed point of the map at $X = 2/3$ lies on the boundary between two neighboring cells, and W has only one non-vanishing diagonal element, namely $W_{11} = 1/2$. On the other hand, if K is not a multiple of 3, the fixed point at 2/3 lies inside cell C_l, where $(l-1)/K < 2/3 < l/K$. Therefore, W has in this case, two non-vanishing diagonal elements, namely $W_{11} = W_{ll} = 1/2$. Thus, while $H(j, 0; j, 0) = 1$ by definition, for $n \geqslant 1$ we have

$$H(j, 0; j, n) = \begin{cases} 2^{-n} \delta_{j,1} & (K = \text{mod } 3) \\ 2^{-n}(\delta_{j,1} + \delta_{j,l}) & (K \neq \text{mod } 3) \end{cases} \tag{5.3}$$

where

$$l = [2K/3] + 1 \tag{5.4}$$

We note that sojourns till time 1, 2,..., etc., do not comprise a set of mutually exclusive events, so that $\{H(j, 0; j, n)\}$ does not represent a probability distribution in n.

Turning to the intermittent case represented by the maps S and A, and considering as before the two-cell partitioning of the interval, we must compute the corresponding sojourn probabilities directly from Eq. (5.1). The multiple integrals involved may be evaluated, like the recurrence time distributions, along the lines indicated in the Appendix. For the map A, we find

$$H_A(L, 0; L, n) = H_A(R, 0; R, n) = 4u_n, \qquad n \geqslant 1 \tag{5.5}$$

with $\{u_r\}$ defined by Eq. (4.7) as before. For the symmetric map S, we get

$$H_S(L, 0; L, n) = \tfrac{16}{3} u_{n+1}, \qquad n \geqslant 1 \tag{5.6}$$

Recalling the asymptotic behavior of u_n [Eq. (4.14)], we may therefore conclude that a slow power-law decay ($\sim n^{-1}$) of the sojourn probability is a characteristic signature of the type of intermittency caused by the tangency of the map to the bissectrix. This is further borne out by the sojourn probability $H_S(R, 0; R, n)$, which is found to be

$$H_S(R, 0; R, n) = 16(-1)^n \bar{\lambda}(v_{n+2} - v_{n+1}), \qquad n \geqslant 1 \tag{5.7}$$

where $\{v_r\}$ has been defined in Eq. (4.11). The map S has no point of tangency in the cell R, and $H_S(R, 0; R, n)$ decays rapidly with increasing n like $\exp(-\mu n)$ with $\mu = \ln(\sqrt{2} + 1)$.

5.2. Escape and First Passage Times

The probability that an escape from C_j occurs for the first time at the nth time step is given by

$$F(j, 0; \bar{j}, n) = \mu^{-1}(C_j) \int_{C_j} dX_0 \int_{C_j} dX_1 \cdots \int_{C_j} dX_{n-1} \int_{\bar{C}_j} dX_n \, \rho(X_0) \times \prod_{r=1}^{n} \delta(X_r - f(X_{r-1})) \tag{5.8}$$

where $\bar{C}_j$ is the complement of C_j. It is evident that, in general (and not merely in the case of a Markov chain), we have also

$$F(j, 0; \bar{j}, n) = H(j, 0; j, n-1) - H(j, 0; j, n), \qquad n \geqslant 1 \tag{5.9}$$

in terms of the halting time probabilities defined in the preceding subsection. The normalization of the distribution $F(j, 0; \bar{j}, n)$ is manifest, since $H(j, 0; j, 0) = 1$ by definition.

For a Markov partitioning with transition matrix $\mathbf{W}$, we have therefore

$$F(j, 0; \bar{j}, n) = (W_{jj})^{n-1} - (W_{jj})^{n} \tag{5.10}$$

In the case of the tent map, with a partitioning of the unit interval into K equal cells, Eq. (5.3) yields

$$F(j, 0; \bar{j}, n) = \begin{cases} 2^{-n}\delta_{j,1} & (K = \text{mod } 3) \\ 2^{-n}(\delta_{j,1} + \delta_{j,l}) & (K \neq \text{mod } 3) \end{cases} \tag{5.11}$$

where $l = [2K/3] + 1$, and $F(j, 0; \bar{j}, n) = \delta_{n,1}$ for all other values of j. The mean escape time out of cells C_1 and C_l (when applicable) is therefore equal to 2, with a variance that is also equal to 2. For all other cells C_j ($j \neq 1$ and $j \neq l$, where applicable), the escape time is equal to 1, with no dispersion.

In the intermittent cases (maps S and A), with just two cells L and R, substitution of Eqs. (5.5)–(5.7) in Eq. (5.9) leads at once to the following expressions:

For the map A,

$$F_A(L, 0; R, n) = F_A(R, 0; L, n) = 4(u_{n-1} - u_n) \tag{5.12}$$

For the map S,

$$F_S(L, 0; R, n) = \tfrac{16}{3}(u_n - u_{n+1}) \tag{5.13}$$

whereas

$$F_S(L, 0; R, n) = 16(-1)^{n-1}\,\bar{\lambda}(v_{n+2} - v_n) \tag{5.14}$$

Notice that these expressions are also the $L \to R$ and $R \to L$ first passage time probabilities, because $\bar{L} = R$, $\bar{R} = L$. Given the asymptotic behavior of v_n [Eq. (4.15)], it follows that $F_S(R, 0; L, n) \sim \exp(-\mu n)$ with $\mu = \ln(\sqrt{2} + 1)$, leading to a mean first passage time

$$\langle n_{RL} \rangle_S = 1.739 \tag{5.15}$$

In contrast, the asymptotic behavior of u_n [Eq. (4.14)] shows that the probabilities in Eqs. (5.12) and (5.13) behave asymptotically like n^{-2}. The corresponding mean escape times diverge logarithmically. This is an important hallmark of the underlying intermittency. To make the point even more explicit, we consider, instead of the partitioning of the phase space $[-1, 1]$ into two halves L and R, a partitioning into $(L_\varepsilon, \bar{L}_\varepsilon)$, where

$L_\varepsilon = [-1, -1+\varepsilon]$, with $0 < \varepsilon \leqslant 1$ [thus $\varepsilon = 1$ corresponds to the original (L, R) partitioning]. Tuning ε helps us understand how the tangency of the maps with the bissectrix at $X = -1$ affects the escape time distribution. Computing the latter quantity, we find (for $n \geqslant 1$)

$$F_A(L_\varepsilon, 0; \bar{L}_\varepsilon, n) = \frac{4}{\varepsilon}(u_{n-1} - u_n) \tag{5.16}$$

and

$$F_S(L_\varepsilon, 0; \bar{L}_\varepsilon, n) = \frac{4}{\varepsilon(1-\varepsilon/4)}(u_n - u_{n+1}) \tag{5.17}$$

where $u_r = u_{r-1}(1 - u_{r-1})$, as before, but now $u_0 = \varepsilon/4$ rather than 1/4. The leading behavior of the distributions in Eqs. (5.16) and (5.17) is $\sim\varepsilon$ for $n\varepsilon \leqslant 1$, and $\sim 1/(n^2\varepsilon)$, for $n\varepsilon \gg 1$. The mean time of escape from L_ε is formally $(4/\varepsilon)\sum_1^\infty u_n$ for the map A, with an extra factor $(1-\varepsilon/4)^{-1}$ in the case of the map S; the divergence of this quantity is thus logarithmic for all $\varepsilon > 0$, however small.

Some remarks are in order here: focusing on the map S, one may wonder whether there are any contradictions among the following results: the *first passage* time n_{RL} has finite moments of all orders [cf. Eq. (5.14)], but the *recurrence* time n_{RR} has only a finite first moment [cf. Eqs. (4.13) and (4.17)]. The divergence of the second and higher moments of n_{RR} is clearly an effect of the exceedingly slow decay of the sojourn time probability in L in the vicinity of the marginally stable fixed point at -1. Even more dramatic, apparently, is the divergence of the first moment itself of the *escape* time n_{LR} out of L_ε [cf. Eqs. (5.13) and (5.17)], while the *recurrence* time n_{LL} has finite moments of all orders.

Once again, the rationale behind this is the fact that the moments of n_{LR} are computed with the probability

$$P(L, 0; L, 1;...; L, n-1; R, n)$$

as the weight factor, while those of n_{LL} are computed with the probability

$$P(L, 0; R, 1;...; R, n-1; L, n)$$

as the weight factor; and we have seen that the halting time probability in R decays exponentially with time, unlike the n^{-1} decay of that in L. Actually these conclusions can be recovered, qualitatively, by taking the continuous time limit, which is legitimate as long as the dynamics is restricted in the vicinity of the marginally stable point.

So far first passage times have only been discussed for the two-cell partition of the intermittent maps. For the sake of completeness, we finally record the first passage time distribution $F(j, 0; k, n)$ in the case of the tent map with a K-cell Markov partitioning of the unit interval. Formally, we have

$$F(j, 0; k, n) = \mu^{-1}(C_j) \int_{C_j} dX_0 \int_{\bar{C}_k} dX_1 \cdots \int_{\bar{C}_k} dX_{n-1} \times \int_{C_k} dX_n \, \rho(X_0) \prod_{r=1}^{n} \delta(X_r - f(X_{r-1})) \tag{5.18}$$

where $n \geqslant 1$, and $\bar{C}_k$ is the complement of C_k. As in the case of the recurrence time distribution, the renewal equation[10] comes to our aid. The generating function

$$\tilde{F}_{jk}(s) = \sum_{n=1}^{\infty} F(j, 0; k, n) \, s^n \tag{5.19}$$

is related to that of $P(j, 0; k, n)$ in a manner analogous to Eq. (3.8): we have

$$\tilde{F}_{jk}(s) = \tilde{P}_{jk}(s)/[1 + \tilde{P}_{kk}(s)] \tag{5.20}$$

As we saw earlier [Eq. (3.10)]

$$\tilde{P}_{jk}(s) = \frac{s}{K(1-s)} + \tilde{R}_{jk}(s) \tag{5.21}$$

in the neighborhood of $s = 1$, where $\tilde{R}_{jk}(s)$ is regular at $s = 1$. Using this in Eq. (5.20), it is easy to establish that the mean time of first passage from C_j to C_k is given by

$$\langle n_{jk} \rangle = K[1 + \tilde{R}_{kk}(1) - \tilde{R}_{jk}(1)] \tag{5.22}$$

6. CONCLUDING REMARKS

In this paper we have analyzed the dynamics of transitions between the cells of a finite phase-space partition in a variety of systems giving rise to chaotic behavior, with special emphasis on the statistics of recurrence times and related quantities. In the case of one-dimensional piecewise linear mappings giving rise to fully developed chaos a Markovian phase space partition could be introduced thanks to which the recurrence problem could be cast into a renewal process. A full theory of transition time statistics of such systems has been worked out, whose main conclusions are as follows: the transition time distributions are exponential, the mean recurrence times are inversely proportional to the cell size and their variance is inversely proportional to the square of the cell size. As a byproduct the dispersion of recurrence times is seen to be quite large, comparable to the mean value itself.

The case of one-dimensional maps giving rise to intermittent behavior turned out to be much more intricate. We have established that, at least for certain phase space partitions, the transitions between cells define a non-Markovian process that cannot be cast into a renewal process. The analysis of the asymptotic (long time) behavior of the transition probabilities revealed the presence of power law distributions, entailing that the variances—and in some cases even the mean values of the corresponding variables—are divergent.

Our results bring out the complexity of recurrence or, more generally, of transition time dynamics in chaotic systems. Thanks to the relative simplicity of the models considered a detailed analysis could nevertheless be carried out, which would have been impossible in systems considered traditionally in statistical mechanics. The results are suggestive, and it may be hoped that they will stimulate new approaches—at least by means of numerical experimentation—to certain real-world problems that have remained elusive so far.

An obvious extension of the present study is to work out a transition time statistics of multivariate systems. Of special importance for the applications would be the extension of the theory to *continuous time* dynamical systems. In this context, an interesting question is to identify the scaling law relating the mean recurrence times to the size of the phase space cell considered and, perhaps also, to the quantifiers of the local dynamics such as attractor dimensions and/or Lyapunov exponents.

APPENDIX. RECURRENCE TIME DISTRIBUTIONS FOR THE MAPS *S* AND *A*

We consider first the discontinuous map A, given by

$$f_A(X) = \begin{cases} 1 - 2(-X)^{1/2}, & X \in L \\ 2X^{1/2} - 1, & X \in R \end{cases} \tag{A.1}$$

as this case is a little less involved than that of the map S, because it has an obvious $L \leftrightarrow R$ symmetry and, moreover, its invariant density is a constant. Owing to the symmetry referred to, the recurrence time distributions

$F_A(L, 0; L, n)$ and $F_A(R, 0; R, n)$ are identical. When $n=1$, it is trivial to see that $F_A(L, 0; L, 1)=P_A(L, 0; L, 1)=3/4$. For $n \geqslant 2$, we have

$$F_A(L, 0; L, n)=\int_{-1}^{0} dX_0 \int_{0}^{1} dX_1 \cdots \int_{0}^{1} dX_{n-1} \int_{-1}^{0} dX_n \times \prod_{r=1}^{n} \delta(X_r - f_A(X_{r-1})) \tag{A.2}$$

The δ-functions have support on the set $X_r = 2X_{r-1}^{1/2} - 1$, or $X_{r-1} = (1+X_r)^2/4$, for $2 \leqslant r \leqslant n$. Defining the monotonically increasing sequence $\{z_r\}$ by the recursion relation

$$z_r = (1+z_{r-1})^2/4, \qquad z_0 = 0 \tag{A.3}$$

we see that the δ-functions successively restrict the range of X_{n-1} to $[z_0, z_1]$, that of X_{n-2} to $[z_1, z_2]$, and so on. We thus obtain

$$\begin{aligned} F_A(L, 0; L, n) &= \int_{-1}^{0} dX_0 \int_{z_{n-2}}^{z_{n-1}} dX_1\, \delta(X_1 - f_A(X_0)) \\ &= \tfrac{1}{4}[(1-z_{n-2})^2 - (1-z_{n-1})^2] \end{aligned} \tag{A.4}$$

remembering that $X_0 \in L$. Using the relation $z_{r-1}^2 = 4z_r - 2z_{r-1} - 1$ to eliminate quadratic terms, we get

$$F_A(L, 0; L, n) = 2z_{n-1} - z_n - z_{n-2}, \qquad n \geqslant 2 \tag{A.5}$$

The limit point of the sequence $\{z_r\}$ is 1. Let us therefore set

$$z_r = 1 - 4u_r \qquad \text{or} \qquad u_r = (1 - z_r)/4 \tag{A.6}$$

Then $\{u_r\}$ is given by

$$u_r = u_{r-1}(1 - u_{r-1}), \qquad u_0 = 1/4 \tag{A.7}$$

which is just the logistic map at parameter value corresponding to marginal stability of the fixed point at the origin. In terms of $\{u_r\}$, we have finally the expression quoted in Eq. (4.8), namely,

$$\begin{aligned} F_A(L, 0; L, n) = F_A(R, 0; R, n) &= 4(u_{n-2}^2 - u_{n-1}^2) \\ &= 4(u_{n-2} - 2u_{n-1} + u_n) \end{aligned} \tag{A.8}$$

We note that this result is valid for $n=1$ as well, on identifying u_{-1} with 1/2, in accordance with Eq. (A.7).

Next, we turn to the symmetric, continuous map S given by

$$f_S(X) = \begin{cases} 1-2(-X)^{1/2}, & X \in L \\ 1-2X^{1/2}, & X \in R \end{cases} \tag{A.9}$$

The $L \leftrightarrow R$ symmetry is no longer present. It is easily seen that

$$\begin{aligned} F_S(L, 0; L, 1) &= P_S((L, 0; L, 1) = 13/16 \\ F_S(R, 0; R, 1) &= P_S(R, 0; R, 1) = 7/16 \end{aligned} \tag{A.10}$$

Let us consider $F_S(R, 0; R, n)$ first. Since $\rho_S(X) = (1-X)/2$, we have $\mu_S(R) = 1/4$. Therefore, for $n \geqslant 2$,

$$\begin{aligned} F_S(R, 0; R, n) = 2 \int_0^1 dX_0(1-X_0) \int_{-1}^0 dX_1 \cdots \int_{-1}^0 dX_{n-1} \\ \times \int_0^1 dX_n \prod_{r=1}^n \delta(X_r - f_S(X_{r-1})) \end{aligned} \tag{A.11}$$

The δ-functions now have support on the set $X_r = 1-2(-X_{r-1})^{1/2}$, or $X_{r-1} = -(1-X_r)^2/4$, for $2 \leqslant r \leqslant n$. Therefore, defining the sequence $z_r = -(1-z_{r-1})^2/4$, $z_0 = 0$, we see that the δ-functions successively imply the range of integration $[z_r, z_{r-1}]$ for X_r, where $r = 1, 2, ..., n-1$. Bearing in mind that $X_0 \in R$, we find

$$\begin{aligned} F_S(R, 0; R, n) &= 2 \int_{-z_{n-1}}^{-z_n} dX_0(1-X_0) \\ &= 4(z_{n-1} - 2z_n + z_{n+1}) \end{aligned} \tag{A.12}$$

after eliminating quadratic terms with the help of the recursion relation. As $\lim_{n \to \infty} z_n = -1$, we set $Z_r = -1 + 4u_r$, to get the expression quoted in Eq. (4.9), namely

$$F_S(R, 0; R, n) = 16(u_{n-1} - 2u_n + u_{n+1}) \tag{A.13}$$

where $\{u_r\}$ is given by Eq. (A.7) as before. It is easily verified that this expression holds good also for $n = 1$.

Next, we evaluate $F_S(L, 0; L, n)$. Since $\mu_S(L) = 3/4$, we have

$$\begin{aligned} F_S(L, 0; L, n) = \tfrac{2}{3} \int_{-1}^0 dX_0(1-X_0) \int_0^1 dX_1 \cdots \int_0^1 dX_{n-1} \int_0^1 dX_n \\ \times \prod_{r=1}^n \delta(X_r - f(X_{r-1})) \end{aligned} \tag{A.14}$$

The δ-functions now have support on the set $X_{r-1}=(1-X_r)^2/4$, $2\leqslant r\leqslant n$. Defining $z_r=(1-z_{r-1})^2/4$, $z_0=1$, we see that the successive ranges of integration are $z_2\leqslant X_{n-1}\leqslant z_0$, $z_1\leqslant X_{n-2}\leqslant z_3$, etc. Carrying the calculation through, we obtain

$$F_S(L, 0; L, n)=\frac{(-1)^n}{3}(z_{n+1}-z_{n-1})(z_{n+1}+z_{n-1}+2) \qquad \text{(A.15)}$$

for $n=1, 2,\ldots$. The presence of the factor $(-1)^n$ is noteworthy. It arises because the sequence $\{z_r\}$ approaches the limit point $3-2\sqrt{2}\equiv\bar{\lambda}^2$ (the fixed point of the map S) in an *oscillatory* manner. Setting $z_r=\bar{\lambda}^2+4\bar{\lambda}v_r$ (where $\bar{\lambda}=\sqrt{2}-1$), we have

$$v_r=-\bar{\lambda}v_{r-1}(1-v_{r-1}), \qquad v_0=1/2 \qquad \text{(A.16)}$$

Substitution in Eq. (A.15) leads (after elimination of the quadratic terms) to

$$F_S(L, 0; L, n)=\frac{16\bar{\lambda}}{3}(-1)^n(v_{n+2}+v_{n+1}-v_n-v_{n-1}) \qquad \text{(A.17)}$$

as stated in Eq. (4.10).

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